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Canonical quantization of a massive particle on AdS_3

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Abstract: The classical theory for a massive free particle moving on the group manifold $AdS_3 \cong \mathrm{SL}(2,\mathbb{R})$ is analysed in detail. In particular a symplectic structure and two different sets of canonical coordinates are explicitly found, corresponding to the Cartan and Iwasawa decomposition of the group. Canonical quantization is performed in two different ways; by imposing the future-directed constraint before and after quantization. It is found that this leads to different quantum theories. The Hilbert space of either theory decomposes into the sum of certain irreducible representations of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$; however, depending on how the constraint is imposed we get different representations. Quantization of the mass occurs, although a continuum exists in the unconstrained theory corresponding to particles that can reverse their direction in time. A quantization in terms of the "chiral" variables of the theory is also carried out giving the same results. Comparisons are made between QFT in AdS_3 and the quantum mechanics derived, and it is found that one of the quantum theories is consistent with the Breitenlohner-Freedman bound.

Keywords: Field Theories in Lower Dimensions, Sigma Models, Conformal Field Models in String Theory, Integrable Field Theories.

Contents

1. Introduction

Our present understanding of string theory in curved backgrounds is rather limited. One of the simplest non-trivial backgrounds would appear to be AdS_3; this space is actually the group manifold $\mathrm{SL}(2,\mathbb{R})$, therefore this allows one to re-express bosonic string theory with a B field as an $\mathrm{SL}(2,\mathbb{R})$ WZW model, which arguably is easier to study. It is important to understand strings in AdS_3 from the point of view of the AdS/CFT correspondence, as a detailed understanding of the string theory would allow non-trivial tests of the conjecture and maybe even insights into its proof. Now, much work has been done on the subject, for example [1]–[3], however a systematic canonical treatment is lacking. An obvious starting point for all this is to study the simpler case of a free particle in such a space, since it should correspond to the $\alpha' \to 0$ limit of the string theory, and of course it is interesting in its own right. The particle lagrangian will be cast in a form similar to the WZW model, and then analyzed in a group theoretic way, following [4, 5].

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In this paper we first study the classical theory for a massive free particle moving on the group manifold $AdS_3 \cong \mathrm{SL}(2,\mathbb{R})$. We will make use of the elegant approach to phase space and Poisson brackets introduced by Witten [6] and Zuckerman [7]. This involves the identification of the phase space with the manifold of all classical solutions, and defining a symplectic form on this manifold directly from the lagrangian. This leads to interesting (quadratic) Poisson brackets in terms of certain "natural" variables. As expected it is found that the current algebras provide a representation of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ the Lie algebra of the isometry group of the manifold. Quantization of this system can be done canonically, leading to the result that the Hilbert space of the theory furnishes the direct sum of certain irreducible representations of the quantum current algebra which is still $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. The irreducible representations turn out to be most of both discrete series, but the exact representations seem to depend on how one imposes the constraint that the particle should be future directed. The consequences of this are that the particle mass becomes quantized (as expected since there is a closed time-like direction). Interestingly C_j^0 in the exceptional interval appears in the unconstrained system leading to a continuum of mass states which enjoy the property of being able to reverse their direction in time. We must note that quantization of a particle on AdS_3 has already been attempted [8], see also [14], however our results differ slightly; if we impose the constraint before quantization we do indeed reproduce their results, however quantizing the unconstrained theory we get a quantum correction to the Casimir which changes the allowed representations, and hence the masses, even after the constraint is imposed in the quantum theory.

A more interesting quantization, going back to Faddeev et al. [9, 5] and later Goddard et al. [4, 10], is carried out too, involving quantities which correspond to the "chiral" nature of the theory inherited from the bivariance of the metric. This gives the same results, as one of our quantum theories, and provides a strategy which will be useful for quantization of the string.

In the final section we compare known results of QFT in AdS_3 [11], namely the Wightman function, to the propagator in quantum mechanics. The propagator can take two different forms, depending on which expression for the Casimir is chosen. The Casimir with the quantum correction leads to a simplification in the formulae and thus appears critical in some sense; further, it is also consistent with the Breitenlohner-Freedman bound in AdS_3.

2. The classical theory

2.1 The geometry of $\mathrm{SL}(2,\mathbb{R})$

The Lie algebra $sl(2,\mathbb{R})$ consists of all real two dimensional traceless matrices. A convenient basis is given by,

$$\mathsf{T}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{T}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{T}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

which satisfy $\mathsf{T_a T_b} = \eta_{ab}1 + \epsilon_{ab}{}^{c}\mathsf{T_c}$, where $(\eta_{ab}) = \mathrm{diag}(1,-1,1)$ and ϵ_{abc} is the usual alternating symbol ($\epsilon_{123} = 1$), and $\epsilon_{ab}{}^{c} = \epsilon_{abd}\eta^{dc}$. Note that the structure constants in this basis are $2\epsilon_{ab}{}^c$. The Killing form of a Lie algebra $\mathfrak{g}$ is defined as a $(0,2)$ tensor κ, such that $\kappa(x,y) = \mathrm{tr}[ad_x ad_y]$ where $x, y \in \mathfrak{g}$. It is easy to show that $\kappa(\mathsf{T}_a, \mathsf{T}_b) = 8\eta_{ab}$, and it is this metric (actually $\frac{1}{8}\kappa$) on the Lie algebra which is used to raise and lower Lie algebra indices.

Now, any group element can be written uniquely as $\mathsf{g} = e^{u\mathsf{T}_2}e^{\rho\mathsf{T}_1}e^{v\mathsf{T}_2}$, where $t = u + v, \phi = v - u \in [0, 2\pi)$ and $\rho \geq 0$, this is called the Cartan decomposition. Any (semi-simple) Lie group possesses a natural left and right invariant metric, $G_{\mu\nu} = \frac{1}{2}\,\mathrm{tr}(\mathsf{g}^{-1}\partial_\mu \mathsf{g}\,\mathsf{g}^{-1}\partial_\nu \mathsf{g})$. A calculation then gives,

$$G = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2\,. \tag{2.2}$$

This is the metric for AdS_3 with the cosmological constant $\Lambda = -1$. To get the metric for the universal cover $\widetilde{AdS_3}$ one simply takes t to range over $\mathbb{R}$.

2.2 Massive particles

Let $\mathsf{g} : \mathbb{R} \to \mathrm{SL}(2,\mathbb{R})$ be the curve corresponding to the particle's worldline. The parameter of the curve g will be called the proper time and will be assumed to take all real values since AdS_3 is geodesically complete. The lagrangian we will use is,

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}(\mathsf{g}^{-1}\dot{\mathsf{g}})^2\,. \tag{2.3}$$

Hence we see, from (2.2), that the lagrangian is simply $\mathcal{L} = G_{\mu\nu}\dot{g^\mu}\dot{g^\nu}$, where $G_{\mu\nu}$ is the metric for AdS_3, although it must be supplemented with the mass-shell constraint (see appendix B).

Now, one may wonder why we have chosen to use (2.3) as our particle lagrangian, and not the more familiar $\mathcal{L} = -2m\sqrt{|G_{\mu\nu}\dot{g}^\mu\dot{g}^\nu|}$, where m is the mass; this latter lagrangian is singular (i.e. $\det\,[\frac{\partial^2\mathcal{L}}{\partial\dot{g}^\mu\partial\dot{g}^\nu}] = 0$ and hence one cannot express $\dot{g}^\mu$ in terms of the canonical momenta), which will lead to a constraint (the mass-shell condition) which by itself is not so bad. What complicates matters is that one cannot gauge fix the reparameterisation invariance covariantly (see [12] for a good account); breaking spacetime covariance is undesirable, as after quantization one needs to show that the theory is still covariant. Also it must be noted that the "square-root lagrangian" is actually classically equivalent to $\mathcal{L} = \frac{1}{e}G_{\mu\nu}\dot{g^\mu}\dot{g^\nu} - em^2$, where e is an einbein on $\mathbb{R}$; this lagrangian can be gauge fixed covariantly, e.g. $e \approx 1$,[1] and in fact with this constraint we get that the phase space structure is the same as that of the lagrangian (2.3) with the mass-shell constraint (see appendix B).

It will be useful to find the hamiltonian formulation for our lagrangian system. Firstly, the canonical momenta are $\pi_\mu \equiv \frac{\partial\mathcal{L}}{\partial\dot{g}^\mu} = 2G_{\mu\nu}(g)\dot{g}^\nu$. The canonical hamiltonian is then found to be $H = \frac{1}{4}G^{\mu\nu}\pi_\mu\pi_\nu$. The mass-shell constraint is $\frac{1}{4}G^{\mu\nu}\pi_\mu\pi_\nu + m^2 \approx 0$. Now at this point one is tempted to take the phase space as $T^*(AdS_3)$ with coordinates (g^μ, π_ν), and proceed as usual by defining the standard Poisson bracket, working out the Noether currents and their algebras in preparation for the quantum theory. However this route will turn out to be plagued with pitfalls at the quantization stage, such as operator ordering

[1] The symbol $\approx$ means weakly equal in the language of Dirac, in other words equal modulo the constraints.

ambiguities in expressions involving the metric in a general coordinate system. Also, solving the equations of motion covariantly is not really possible in this formalism. Since we are dealing with a group manifold we will employ a more group theoretic approach.

The left invariance of the metric gives the current $\mathsf{L} = -\dot{\mathsf{g}}\mathsf{g}^{-1}$ and the right invariance gives $\mathsf{R} = \mathsf{g}^{-1}\dot{\mathsf{g}}$. The equations of motion are simply the current conservation laws $\dot{\mathsf{L}} = \dot{\mathsf{R}} = 0$. A general solution can be easily obtained in this language, $\mathsf{g}(\tau) = e^{-\mathsf{L}\tau}\mathsf{g}(0) = \mathsf{g}(0)e^{\mathsf{R}\tau}$.

A more useful form for a general time-like (it is here that restriction to massive particles occurs) solution can be derived as follows. Firstly note that isometries map time-like geodesics into time-like geodesics. The action of the isometries on geodesics is transitive. Hence given one time-like geodesic, $\mathsf{g}(\tau) = e^{p\tau\mathsf{T}_2}$ say ($p > 0$ for future-directed), all others are given by the action of an isometry on it. Therefore a general time-like geodesic can be written as $\mathsf{g}(\tau) = \mathsf{u}_0 e^{p\tau\mathsf{T}_2}\mathsf{v}_0$, where u_0 and v_0 are group elements. It is clear that the map $\mathsf{u}_0 \mapsto \mathsf{u}_0\mathsf{h}$ and $\mathsf{v}_0 \mapsto \mathsf{h}^{-1}\mathsf{v}_0$, with $\mathsf{h} = e^{a\mathsf{T}_2}$, leaves $\mathsf{g}(\tau)$ unchanged (in fact for any such map which leaves $\mathsf{g}(\tau)$ unchanged, h must be of this form); therefore,

$$\mathsf{g}(\tau) = \tilde{\mathsf{u}}e^{(q+p\tau)\mathsf{T}_2}\tilde{\mathsf{v}}\,, \tag{2.4}$$

where $\tilde{\mathsf{u}}$ and $\tilde{\mathsf{v}}$ belong to $\mathrm{SL}(2,\mathbb{R})/T$ and $T\backslash\,\mathrm{SL}(2,\mathbb{R})$ respectively, where T is the Cartan subgroup given by $T = \{e^{a\mathsf{T}_2}\} = SO(2)$. Note that using (2.4) the hamiltonian works out to be $H = -p^2 \approx -m^2$.

2.3 Phase space and Poisson brackets

We will follow the elegant approach of Zuckerman [7] and Witten [6], in order to get a covariant derivation of the Poisson brackets. This involves defining the phase space $\mathcal{S}$, as the manifold of classical solutions of the Euler-Lagrange equations of the lagrangian. Then in order to define Poisson brackets, as is well known, one needs to find a symplectic form on $\mathcal{S}$; recall this is simply a closed non-degenerate two form. The idea is that one can avoid moving into the hamiltonian formalism by defining a symplectic form directly from the lagrangian $\mathcal{L}$; we sketch how this works in the general case of a one-dimensional field theory. Consider a theory with fields $\phi : \mathbb{R} \to M$ and an action $S(\phi) = \int_{\mathbb{R}} \mathcal{L}(\phi)$. Let $\mathcal{F}$ be the space of fields ϕ and $\mathcal{S}$ the submanifold of solutions to the variational problem $\delta S(\phi) = 0$, where δ is the exterior derivative on $\mathcal{F}$. Now, $\mathcal{L}$ is a (0,1)-form on $\mathcal{F} \times \mathbb{R}$; it can be shown that $\delta\mathcal{L} = E + d\theta$ where d is the exterior derivative on $\mathbb{R}$ and the decomposition is unique. Then one defines the two form (on $\mathcal{S}$) as $\omega = \delta\theta(\tau)$ which is the symplectic structure desired; it is clear, since d and δ anticommute on $\mathcal{F} \times \mathbb{R}$, and that E vanishes on $\mathcal{S} \times \mathbb{R}$ (this corresponds to the Euler-Lagrange equations as $\delta S(\phi) = \int_{\mathbb{R}} E(\phi, \delta\phi)$) that $d\omega = 0$ showing that no special choice of τ has been made.

Applying this procedure to (2.3) gives $\theta(\tau) = -\,\mathrm{tr}[\delta\mathsf{g}\;\partial_\tau(\mathsf{g}^{-1})]$. Therefore a symplectic form for $\mathcal{S}$ is given by,

$$\omega = -\frac{1}{2}\,\mathrm{tr}\left(\delta\mathsf{g} \wedge \delta(\partial_\tau\mathsf{g}^{-1})\right), \tag{2.5}$$

where the factor of two has been introduced for convenience.

Poisson brackets for the theory are defined in the usual way,

$$\{f, g\} = \omega^{ab}\partial_a f\,\partial_b g, \qquad \text{where} \quad f, g \in C^\infty(\mathcal{S}) \qquad \text{and} \quad \omega_{ac}\omega^{cb} = \delta^a_b. \tag{2.6}$$

Now, one can choose a parameterisation of the cosets such that $\tilde{\mathsf{u}} = e^{A\mathsf{T}_2}e^{B\mathsf{T}_1}$ and $\tilde{\mathsf{v}} = e^{\bar{B}\mathsf{T}_1}e^{\bar{A}\mathsf{T}_2}$. Note that choosing a different parameterisation simply redefines q appearing in (2.4). Using this parameterisation, and substituting into (2.5) we get,

$$\omega = \delta\lambda\,, \qquad \text{where} \qquad \lambda = p\delta q + M\delta A + \bar{M}\delta\bar{A} \tag{2.7}$$

and $M = p\cosh 2B$ and $\bar{M} = p\cosh 2\bar{B}$. Hence we have found canonical coordinates on the phase space with only the following non-zero Poisson brackets,

$$\{q,p\} = \{A,M\} = \{\bar{A},\bar{M}\} = 1\,. \tag{2.8}$$

It is also interesting to work out Poisson brackets of other variables. Define $\mathsf{u} = \tilde{\mathsf{u}}e^{q\mathsf{T}_2}$ and $\mathsf{v} = e^{q\mathsf{T}_2}\tilde{\mathsf{v}}$, and use the following convenient notation [2] for $\mathsf{a} \in \mathrm{Mat}(2,\mathbb{R})$, $\mathsf{a}_1 = \mathsf{a}\otimes 1$ and $\mathsf{a}_2 = 1\otimes\mathsf{a}$. Then it is found that,

$$\begin{aligned}
\{\tilde{\mathsf{u}}_1,\tilde{\mathsf{v}}_2\} &= \{\tilde{\mathsf{u}},p\} = \{\tilde{\mathsf{v}},p\} = 0 &(2.9)\\
\{\mathsf{u}_1,\mathsf{u}_2\} &= \mathsf{u}_1\mathsf{u}_2\mathsf{r}_{12} &(2.10)\\
\{\mathsf{v}_1,\mathsf{v}_2\} &= -\mathsf{r}_{12}\mathsf{v}_1\mathsf{v}_2 &(2.11)\\
\mathsf{r}_{12} &= \frac{1}{2p}(\mathsf{T}_1\otimes\mathsf{T}_3 - \mathsf{T}_3\otimes\mathsf{T}_1)\,. &(2.12)
\end{aligned}$$

Now, the currents are easily found to be,

$$\begin{aligned}
\mathsf{L} &= -\tilde{\mathsf{u}}p\mathsf{T}_2\tilde{\mathsf{u}}^{-1} = -\mathsf{u}p\mathsf{T}_2\mathsf{u}^{-1}\\
\mathsf{R} &= \tilde{\mathsf{v}}^{-1}p\mathsf{T}_2\tilde{\mathsf{v}} = \mathsf{v}^{-1}p\mathsf{T}_2\mathsf{v}
\end{aligned} \tag{2.13}$$

which then give the following brackets

$$\{\mathsf{L}_1,\mathsf{u}_2\} = -\mathcal{C}_{12}\mathsf{u}_2\,, \qquad \{\mathsf{R}_1,\mathsf{v}_2\} = \mathsf{v}_2\mathcal{C}_{12}\,, \tag{2.14}$$

where $\mathcal{C}_{12} = \mathsf{T}^a\otimes\mathsf{T}_a$ is the tensor Casimir. This gives us the current algebras,

$$\{\mathsf{L}_1,\mathsf{L}_2\} = [\mathsf{L}_2,\mathcal{C}_{12}]\,, \qquad \{\mathsf{R}_1,\mathsf{R}_2\} = [\mathsf{R}_2,\mathcal{C}_{12}]\,. \tag{2.15}$$

Since L and R belong to the Lie algebra $sl(2,\mathbb{R})$, then we can write the current algebras in terms of their components in the basis $\{\mathsf{T}_a\}$. We get,

$$\{L_a,L_b\} = 2\epsilon_{ab}{}^{c}L_c\,, \qquad \{R_a,R_b\} = 2\epsilon_{ab}{}^{c}R_c \tag{2.16}$$

which are simply two copies of $sl(2,\mathbb{R})$. It should be noted that the current algebras can be deduced from equations (2.9)–(2.12) alone without the need to resort to canonical coordinates. A possible sticking point in the derivation of (2.14) appears to be when one gets to,

$$\{\mathsf{L}_1,\mathsf{u}_2\} = -\mathsf{u}_1\mathsf{u}_2\mathcal{C}_{12}\mathsf{u}_1^{-1}\,. \tag{2.17}$$

[2] Define $(\mathsf{a}\otimes\mathsf{b})_{ik,jl} = (\mathsf{a})_{ij}\,(\mathsf{b})_{kl}$, and $(\mathsf{u})_{ik,pq}(\mathsf{v})_{pq,jl} = (\mathsf{u}\,\mathsf{v})_{ik,jl}$. Then we get nice laws such as $(\mathsf{a}\otimes\mathsf{b})(\mathsf{c}\otimes\mathsf{d}) = \mathsf{ac}\otimes\mathsf{bd}$.

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It seems that the only way to know how $\mathcal{C}_{12}$ moves past the u's is to use an explicit parameterisation (and hence canonical coordinates). In fact we can avoid working in canonical coordinates as follows. Consider $\mathbb{P} = \frac{1}{2}(\mathsf{I} \otimes \mathsf{I} + \mathcal{C}_{12})$. It is convenient to use the isomorphism $\mathrm{Mat}(n, \mathbb{R}) \otimes \mathrm{Mat}(n, \mathbb{R}) \cong \mathrm{Mat}(n^2, \mathbb{R})$ defined by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix} \tag{2.18}$$

for $n = 2$. Then it is easy to show that,

$$\begin{aligned} \mathcal{C}_{12} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathbb{P} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \end{aligned} \tag{2.19}$$

Note that $\mathbb{P}^2 = \mathsf{I} \otimes \mathsf{I}$. If we consider $\mathsf{a} \in \mathrm{Mat}(2, \mathcal{A})$ where $\mathcal{A}$ is a not necessarily commutative algebra, then it is a straightforward computation to check that,

$$\mathbb{P}\mathsf{a}_1\mathsf{a}_2 = \mathsf{a}_2\mathsf{a}_1\mathbb{P} \tag{2.20}$$

which reduces to $\mathcal{C}_{12}\mathsf{a}_1\mathsf{a}_2 = \mathsf{a}_1\mathsf{a}_2\mathcal{C}_{12}$ in the case where $\mathcal{A}$ is commutative. Applying this to (2.17) (where $\mathcal{A} = \mathbb{R}$) we get the desired result. Equation (2.20) will be needed in the quantum theory.

2.4 Global coordinate systems and constraints

In the previous section we parameterised the cosets using the Cartan decomposition, leading to a global set of coordinates $(q, p, A, M, \bar{A}, \bar{M})$ for the phase space. Actually, this is not quite true since A and $\bar{A}$ are identified modulo 2π, and hence as functions $S^1 \to \mathbb{R}$ are not continuous and hence not smooth. This corresponds to the well known fact that one cannot cover a circle with a single chart. Thus, strictly, the formulas involving these coordinates at best are valid in some open region of the circles, for example everywhere except one point, such as 2π. A set of Poisson brackets which do hold globally are $\{\sin A, M\} = \cos A$ and $\{\cos A, M\} = -\sin A$, and these serve as a global replacement of $\{A, M\} = 1$, which as we have explained can only be satisfied at best everywhere except at a point. Of course all the formulas derived actually involve only $\sin A$ or $\cos A$ and hence are all valid globally. See Isham [13] for a good account of such subtleties.

Now it is clear that the future-directed constraint $p > 0$ will need to be imposed. This can be either done before or after quantization. In the coordinates developed so far this is best done after. This is because restricting the phase space classically to $p > 0$, which implies $M > 0$ and $\bar{M} > 0$, means that we will have to quantize canonical variables of the form $S^1 \times \mathbb{R}^+$ which is problematic. Instead we will derive a set of coordinates which after the restriction $p > 0$ still allow an easy quantization.

To do this, we need the Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$. This tells us that any element of the group manifold can be written as $\mathsf{g} = e^{A\mathsf{N}_+} e^{B\mathsf{T}_3} e^{C\mathsf{T}_2}$, where $\mathsf{N}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and thus the coset G/H can be parameterised by $\tilde{\mathsf{u}} = e^{A\mathsf{N}_+} e^{B\mathsf{T}_3}$. If one proceeds to calculate the symplectic form, we get the same expression as before except that $M = -\frac{1}{2} p e^{-2B}$ and similarly for $\bar{M}$. However, now the variables $A, \bar{A}$ are not periodic but take all real values. This means that if one imposes the constraint classically the phase space restricts to three canonical pairs of the form $\mathbb{R}^+ \times \mathbb{R}$ or $\mathbb{R}^- \times \mathbb{R}$ which are both straightforward to deal with since both these are symplectomorphic to the standard $\mathbb{R} \times \mathbb{R}$ phase space. More explicitly, one finds that the current $\mathsf{L} = -\dot{\mathsf{g}}\mathsf{g}^{-1}$ has the following components in these Iwasawa coordinates,

$$
\begin{aligned}
L^3 &= 2AM \\
L^+ &= L^1 - L^2 = -\left(\frac{p^2}{2M} + 2A^2 M\right) \\
L^- &= L^1 + L^2 = 2M
\end{aligned}
\tag{2.21}
$$

and of course it is easy to verify from the canonical Poisson brackets that they satisfy the $sl(2, \mathbb{R})$ algebra. Analogous expressions hold for the right current R.

3. The quantum theory

3.1 Canonical quantization

Canonical quantization has a long history and many problems. Schematically, given a phase space $\mathcal{S}$ and canonical coordinates (q_i, p_i), then the quantum theory is constructed via a correspondence map $\hat{}\,: C^\infty(\mathcal{S}) \to \mathrm{End}\, \mathcal{H}$, such that the quantum observables are hermitean with respect to the inner product on $\mathcal{H}$. The canonical coordinates will satisfy $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, and

$$
\lim_{\hbar \to 0} \frac{[\hat{f}, \hat{g}]}{i\hbar} = \widehat{\{f, g\}}
$$

for more general observables. Now this only works if the canonical coordinates take all real values, and is not particularly useful unless they are global coordinates for the phase space. It is also clear that ordering ambiguities arise upon quantization, and in general these have to be resolved case by case.

Before moving on we recall the standard, straightforward example of when the phase space is $\mathbb{R}^n \times \mathbb{R}^n$. Then we take global coordinates (q_i, p^i) and of course the natural symplectic form $\omega = \sum_i dp^i \wedge dq_i$ gives the Poisson brackets $\{q_i, p^j\} = \delta_i^j$. To quantize the system one takes unitary representations of this algebra, called the Heisenberg algebra, and it is well known that there exists a unique irreducible representation of this realised on $L^2(\mathbb{R}^n)$ (in fact the more precise statement concerns unitary representations of the Heisenberg group since this will involve strictly bounded operators, see [13]).

3.1.1 Cartan coordinates

If we carry this procedure out for the system studied in the previous section then we simply get the following quantum conditions,[3]

$$[q,p] = i\hbar, \qquad [e^{iA}, M] = -\hbar e^{iA}, \qquad [e^{i\bar{A}}, \bar{M}] = -\hbar e^{i\bar{A}}. \tag{3.1}$$

The hamiltonian will be $H = -p^2$, and a general observable, O, will evolve according to Heisenberg's equation of motion $i\hbar\dot{O} = [O, H]$, which is equivalent to $O(\tau) = e^{iH\tau/\hbar}O(0)e^{-iH\tau/\hbar}$. Of course we are now interested in the quantum versions of the current algebras; tentatively we take the left current to be $\mathsf{L} = -\tilde{\mathsf{u}}p\mathsf{T}_2\tilde{\mathsf{u}}^{-1} - i\alpha_L\hbar\mathsf{I}$ where α_L is some unknown constant. Now, if we compute the components of the current we find that indeed there are potential ordering problems since we end up with functions of A times functions of M, however they happen to be all hermitean combinations of the canonical variables, and so we conclude that we have taken an acceptable definition for the current. Note that if we choose $\alpha_L = 1$ we get $\operatorname{tr}\mathsf{L} = 0$ which is a desirable property (but not essential). Anyway, we get

$$L_2 = M\,, \qquad L_\pm = -e^{\pm 2iA}\sqrt{(M \pm \hbar)^2 - p^2}\,, \tag{3.2}$$

where $L_\pm = L_3 \pm iL_1$. It is easy to the check that $< L_1, L_2, L_3 >$ provide a unitary representation of the algebra of $sl(2,\mathbb{R})$,

$$[L_2, L_\pm] = \pm 2\hbar L_\pm\,, \qquad [L_+, L_-] = -4\hbar L_2\,. \tag{3.3}$$

The Casimir $Q_L = \eta^{ab}L_aL_b$ can be also calculated to give,

$$Q_L = -p^2 + \hbar^2\,. \tag{3.4}$$

A similar set of calculations for the right algebra gives,

$$Q_R = -p^2 + \hbar^2\,. \tag{3.5}$$

Therefore we deduce that $Q_L = Q_R$ which then tells us that the Hilbert space of states for the quantum theory decomposes as follows,

$$\mathcal{H} = \bigoplus_{\varrho} \mathcal{V}^{\mathcal{L}}{}_{\varrho} \otimes \mathcal{V}^{\mathcal{R}}{}_{\varrho}\,, \tag{3.6}$$

where ϱ is the eigenvalue of the Casimir which labels the irreducible representations, and $\mathcal{V}^{\mathcal{L},\mathcal{R}}{}_{\varrho}$ is the carrier space of an irreducible representation. Explicitly (for the left algebra) we have,

$$\begin{aligned} Q_L|\varrho, m\rangle &= 4\hbar^2\varrho|\varrho, m\rangle \\ L_2|\varrho, m\rangle &= 2\hbar m|\varrho, m\rangle \\ L_\pm|\varrho, m\rangle &= 2\hbar\sqrt{\varrho + m(m \pm 1)}|\varrho, m \pm 1\rangle\,, \end{aligned} \tag{3.7}$$

[3] We make use here of the correct quantization of the angular variables, as discussed in the previous section and in more detail in [13].

where $\{|\varrho, m\rangle\}$ is a basis for $\mathcal{V^L}_\varrho$, and m is either integer or half integer depending on the representation. However not all irreducible representations of $sl(2,\mathbb{R})$ are allowed. In fact the only ones allowed are D_j^+, D_j^- (where $\varrho = -j(j+1)$, $j \in \{-1, -3/2, \ldots\}$) and C_ϱ^0 for $0 < \varrho < \frac{1}{4}$. This follows from the fact that $Q_L < \hbar^2$, and hence $\varrho < \frac{1}{4}$.

We deduce that the allowed values for the mass (in units of $\hbar$) [4] are,

$$\begin{aligned} \mu &= |2j+1|\,, \qquad j \in \{-1, -3/2, \ldots\} \\ \mu &= \sqrt{1-4\varrho}\,, \qquad 0 < \varrho < \frac{1}{4}\,, \end{aligned} \tag{3.8}$$

that is μ is either a positive integer or in the interval $(0, 1)$.

Quantization of the mass is not unexpected since AdS_3 has closed time-like direction, that is topologically it is $S^1 \times \mathbb{R}^2$, where the S^1 refers to time. The continuum $0 < \mu < 1$ is more interesting.

An explicit basis for $\mathcal{H}$ can be constructed from a state $|0\rangle$ satisfying $p|0\rangle = M|0\rangle = \bar{M}|0\rangle = 0$. Namely $\{|\mu, m, \bar{m}\rangle\}$, where

$$|\mu, m, \bar{m}\rangle = e^{i\mu q} e^{2imA} e^{2i\bar{m}\bar{A}} |0\rangle \tag{3.9}$$

will span the Hilbert space for appropriate values of $\mu, m, \bar{m}$. It is in fact an orthogonal basis and normalised if $\langle 0|0\rangle = 1$.

We have not yet imposed the constraint $p > 0$ corresponding to the future directed condition of the geodesics. Now we need to find a way to impose this condition quantum mechanically. This turns out to be actually fairly easy. We merely note that classically the constraint told us that $M > 0$. Hence we see that quantum mechanically a sensible constraint is $L_2 > 0$; note the corresponding statement $R_2 < 0$ also exists. Thus we need to pick a subspace of our Hilbert space for which these conditions hold. This is easy to do and gives,

$$\mathcal{H}_{p>0} = \bigoplus_{j \le -1} D_j^+ \otimes D_j^- \,. \tag{3.10}$$

Note in particular that the mysterious continuum has disappeared. This continuum actually corresponds to states which can flip their direction in time and of course have no classical counterpart; naturally they are excluded in the quantum theory after imposing the future directed constraint.

3.1.2 Iwasawa coordinates

In these coordinates we have the interesting option of imposing the constraint $p > 0$ before or after quantization. Interestingly this leads to slightly different quantum theories.

First we will mimic the previous section and quantize the unconstrained system and then impose the constraint. Of course we hope to reproduce the same results. The quantum conditions are,

$$[q, p] = i\hbar\,, \qquad [A, M] = i\hbar\,, \qquad [\bar{A}, \bar{M}] = i\hbar\,, \tag{3.11}$$

[4] The geodesic $\mathsf{g}(\tau) = e^{(q+p\tau)\mathsf{T}_2}$ in the coordinates $x^\mu = (t, \rho, \phi)$ is $g^\mu(\tau) = (q + p\tau, 0, \phi_0)$. Hence $\dot{g}^\mu = (p, 0, 0)$, and thus $m^2 = \mu^2\hbar^2 \equiv -G_{\mu\nu}\dot{g}^\mu\dot{g}^\nu = p^2$. Of course the mass $\mu\hbar$ is then defined as the positive square root. This is all classical; appendix B tells us that the same holds quantum mechanically i.e. $p^2 = m^2$ on physical states.

and note there are no subtleties due to periodic variables here. Resolving some of the ordering ambiguities in the currents (imposing that the components be self-adjoint suffices) leads to our left current being,

$$
\begin{aligned}
L^3 &= AM + MA \\
L^+ &= L^1 - L^2 = -\left(\frac{p^2}{2M} + 2AMA\right) \\
L^- &= 2M
\end{aligned}
\tag{3.12}
$$

and these satisfy the same $sl(2,\mathbb{R})$ algebra as in the previous section, as they should! The Casimir can be computed, and reassuringly we get $Q_L = -p^2 + \hbar^2$, which is the same as we got by quantization of the system in the very different Cartan coordinates. Note this wasn't a priori guaranteed, since it isn't necessarily obvious that quantizing a system in completely different canonical coordinates is compatible. Of course, from here on we'll get the same quantum theory as in the previous section before and after the future-directed constraint is imposed.

Now, as we've mentioned in these coordinates we can actually quantize the constrained classical system. To do this, we briefly explain how one would quantize the classical system with phase space $\mathbb{R} \times \mathbb{R}^+$, see [13]. Let (x, π) be global coordinates. There is a nice way of mapping this to the standard $\mathbb{R} \times \mathbb{R}$ phase space; simply define the diffeomorphism $(x, \pi) \to (x\pi, \log \pi) \equiv (\tilde{x}, \tilde{\pi})$. It then follows that $\omega = d\pi \wedge dx = d\tilde{\pi} \wedge d\tilde{x}$ showing that the phase spaces are symplectomorphic as previously asserted. Therefore one can choose $(\tilde{x}, \tilde{\pi})$ as canonical coordinates and simply quantize as usual via a Heisenberg algebra, $[\tilde{x}, \tilde{\pi}] = i\hbar$. Note since $\pi = e^{\tilde{\pi}}$ it follows that $[\tilde{x}, \pi] = i\hbar\pi$, which can be taken as our fundamental quantum condition for quantum mechanics on $\mathbb{R} \times \mathbb{R}^+$. Now, lets apply this to our system. Expressing the classical current L in components in the coordinates $(\tilde{q} = qp, p, \tilde{A}, M = -e^{\tilde{M}})$ gives,

$$
\begin{aligned}
L^3 &= 2\tilde{A} \\
L^+ &= L^1 - L^2 = -\left(\frac{p^2}{2M} + \frac{2\tilde{A}^2}{M}\right) \\
L^- &= L^1 + L^2 = 2M\,.
\end{aligned}
\tag{3.13}
$$

Quantization then involves imposing the conditions,

$$
[\tilde{q}, p] = i\hbar p\,, \qquad [\tilde{A}, M] = i\hbar M\,, \qquad [\tilde{\bar{A}}, \bar{M}] = i\hbar \bar{M}\,,
\tag{3.14}
$$

and quantization of the currents requires simply resolving the order ambiguity in L^+ which we do by writing $L^+ = -(\frac{p^2}{2M} + 2\tilde{A}M^{-1}\tilde{A})$. Then, once again it is an easy exercise showing that we have the $sl(2,\mathbb{R})$ algebra satisfied using the quantum conditions. Interestingly computing the Casimir gives $Q_L = -p^2$ in contrast to the previous quantizations. Now, the Hilbert space decomposes into irreducible representations of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$, and it is easy to see that $L_2 > 0$ as an operator (since $M < 0$). Thus since $Q < 0$ (the right algebra will have the same Casimir) we see that the Hilbert space decomposes as,

$$
\mathcal{H}_{p>0} = \bigoplus_{j \le -\frac{3}{2}} D_j^+ \otimes D_j^-\,.
\tag{3.15}
$$

Note this differs from our other quantum theory by the absence of $D^+_{-1} \otimes D^-_{-1}$ in the Hilbert space; of course we get a different mass spectrum, namely $\mu = \sqrt{j(j+1)}$, which agrees with [8].

3.2 "Chiral" quantization

In this section we aim to show how a subtly different kind of quantization can be done, involving quantities which relate more directly to the symmetries of the system. The proposed quantization [9, 5, 4], consists of quantizing q, p to hermitean operators as before, and quantizing u, v to operator matrices which are unitary in the sense $\mathsf{u}_{ab}\mathsf{u}^\dagger_{cb} = \delta_{ac}$.[5] The quantum conditions are

$$[q,p] = i\hbar\,, \qquad [\tilde{\mathsf{u}}_1, \tilde{\mathsf{v}}_2] = 0\,, \qquad [\tilde{\mathsf{u}}, p] = [\tilde{\mathsf{v}}, p] = 0 \tag{3.16}$$

$$\mathsf{u}_1\mathsf{u}_2 = \mathsf{u}_2\mathsf{u}_1\mathsf{B}_{12}\,, \qquad \mathsf{v}_1\mathsf{v}_2 = \mathsf{B}_{12}^{-1}\mathsf{v}_2\mathsf{v}_1\,, \tag{3.17}$$

where $\tilde{\mathsf{u}} = \mathsf{u}e^{-q\mathsf{T}_2}$ and $\tilde{\mathsf{v}} = e^{-q\mathsf{T}_2}\mathsf{v}$. Quantum consistency requires that

$$\mathsf{B} = \mathsf{I}\otimes\mathsf{I} + i\hbar\mathsf{r}_{12} + O(\hbar^2)\,, \qquad \text{Classical limit} \tag{3.18}$$

$$\mathsf{B}^{-1} = \mathsf{B}^\dagger = \mathbb{P}\mathsf{B}\mathbb{P}\,, \qquad \text{Unitarity and antisymmetry} \tag{3.19}$$

$$\mathsf{B}_{23}(\tilde{p}_1)\mathsf{B}_{13}(p)\mathsf{B}_{12}(\tilde{p}_3) = \mathsf{B}_{12}(p)\mathsf{B}_{13}(\tilde{p}_2)\mathsf{B}_{23}(p)\,, \qquad \text{Associativity} \tag{3.20}$$

$$[\mathsf{B}, e^{q((\mathsf{T}_2)_1 + (\mathsf{T}_2)_2)}] = 0 \qquad \text{Locality}\,, \tag{3.21}$$

where $\tilde{p} = p - i\hbar\mathsf{T}_2$ (note that $p\mathsf{u} = \mathsf{u}\tilde{p}$). Equations (3.19) follow from u being unitary and equations (3.17). Equation (3.20) comes from associativity $\mathsf{u}_1(\mathsf{u}_2\mathsf{u}_3) = (\mathsf{u}_1\mathsf{u}_2)\mathsf{u}_3$, where $\mathsf{u}_1 = \mathsf{u}\otimes 1 \otimes 1$ and similarly for the others. Equation (3.21) deserves some attention; it is equivalent to $[\mathsf{g}_1(\tau), \mathsf{g}_2(\tau)] = 0$ hence ensuring locality in the quantum theory. We will show this at $\tau = 0$, the general result follows from Heisenberg's equations of motion. First note that $\mathsf{g}(0) = \tilde{\mathsf{u}}\mathsf{Q}\tilde{\mathsf{v}} = \mathsf{u}\mathsf{Q}^{-1}\mathsf{v} = \mathsf{u}\tilde{\mathsf{v}} = \tilde{\mathsf{u}}\mathsf{v}$, where $\mathsf{Q} = e^{q\mathsf{T}_2}$. Now, the argument runs as follows:

$$\begin{aligned}
\mathsf{g}_1(0)\mathsf{g}_2(0) &= \mathsf{u}_1\tilde{\mathsf{v}}_1\tilde{\mathsf{u}}_2\mathsf{v}_2 = \mathsf{u}_1\tilde{\mathsf{u}}_2\tilde{\mathsf{v}}_1\mathsf{v}_2 = \mathsf{u}_1\mathsf{u}_2\mathsf{Q}_2^{-1}\mathsf{Q}_1^{-1}\mathsf{v}_1\mathsf{v}_2 \\
&= \mathsf{u}_2\mathsf{u}_1\mathsf{B}_{12}\mathsf{Q}_2^{-1}\mathsf{Q}_1^{-1}\mathsf{B}_{12}^{-1}\mathsf{v}_2\mathsf{v}_1 = \mathsf{u}_2\mathsf{u}_1\mathsf{Q}_2^{-1}\mathsf{Q}_1^{-1}\mathsf{v}_2\mathsf{v}_1 \\
&= \mathsf{u}_2\tilde{\mathsf{u}}_1\tilde{\mathsf{v}}_2\mathsf{v}_1 = \mathsf{u}_2\tilde{\mathsf{v}}_2\tilde{\mathsf{u}}_1\mathsf{v}_1 = \mathsf{g}_2(0)\mathsf{g}_1(0)\,,
\end{aligned}$$

where (3.21) was used in the fifth equality and the quantum conditions were used in the other steps. It is easy to see, from the algebra above, that requiring $[\mathsf{g}_1(0), \mathsf{g}_2(0)] = 0$ implies (3.21), hence completing the equivalence.

Now, an explicit formula for B the braiding matrix is wanted. If we start from the ansatz,

$$\mathsf{B}(p) = \exp\left(-\sum_{\alpha\in\{\pm 1\}} \theta_\alpha(p)\mathsf{E}_\alpha \otimes \mathsf{E}_{-\alpha}\right)\,, \tag{3.22}$$

[5]Note that unitarity of u and v is consistent with canonical quantization if, for example, the choices $B^\dagger = -B$ and $\bar{B}^\dagger = -\bar{B}$ are made for the Cartan coordinates. This is allowed since the canonical variables M and $\bar{M}$ are even functions of B and $\bar{B}$ respectively, and hence their Hermiticity is unaffected by such a choice.

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where $\mathsf{E}_\pm = \frac{1}{2}(\mathsf{T}_3 \pm i\mathsf{T}_1)$, and then impose the conditions above we get $\theta_{-\alpha} = -\theta_\alpha$ (from (3.19) and (3.21) independently), and that θ_α can be any analytic function of p or p^{-1} (from (3.20)). A computation gives,

$$\mathsf{B}(p) = \mathsf{I}\otimes\mathsf{I} - \sin^2(\theta/2)(1\otimes 1 + \mathsf{T}_2\otimes\mathsf{T}_2) - \sum_\alpha \sin\theta_\alpha \mathsf{E}_\alpha \otimes \mathsf{E}_{-\alpha}\,, \tag{3.23}$$

where $\theta \equiv \theta_{+1}$. Imposing the classical limit implies,

$$\sin\theta_\alpha = \frac{\hbar}{p\alpha} + O(\hbar^2)\,. \tag{3.24}$$

If we make the choice $\sin\theta_\alpha = \frac{\hbar}{\alpha p}$ then we get,

$$\mathsf{B}(p) = \mathsf{I}\otimes\mathsf{I} + i\hbar\,\mathsf{r}(p) - \frac{1}{2}\left(1 - \sqrt{1 - \frac{\hbar^2}{p^2}}\right)(\mathsf{I}\otimes\mathsf{I} + \mathsf{T}_2\otimes\mathsf{T}_2)\,. \tag{3.25}$$

Now we come to the problems of ordering of the quantum variables. In particular how do we define the currents L, R in the quantum theory. Motivated by the canonical quantization we define the quantum currents to be,

$$\begin{aligned} \mathsf{L} &= -\tilde{\mathsf{u}} p \mathsf{T}_2 \tilde{\mathsf{u}}^{-1} - i\alpha_L \hbar \mathsf{I} \\ \mathsf{R} &= \tilde{\mathsf{v}}^{-1} p \mathsf{T}_2 \tilde{\mathsf{v}} + i\alpha_R \hbar \mathsf{I}\,, \end{aligned} \tag{3.26}$$

where $\alpha_{L,R}$ are unknown constants. This now allows one to calculate the following brackets,

$$\begin{aligned} [\mathsf{L}_1, \mathsf{u}_2] &= -i\hbar\mathcal{C}_{12}\mathsf{u}_2\,, & [\mathsf{R}_1, \mathsf{v}_2] &= i\hbar\mathsf{v}_2\mathcal{C}_{12} \\ [\mathsf{L}_1, \mathsf{L}_2] &= i\hbar[\mathsf{L}_2, \mathcal{C}_{12}]\,, & [\mathsf{R}_1, \mathsf{R}_2] &= i\hbar[\mathsf{R}_2, \mathcal{C}_{12}] \end{aligned} \tag{3.27}$$

which reduce to the algebras of $sl(2,\mathbb{R})$ when one expands $\mathsf{L} = L^a\mathsf{T}_a + k\mathsf{I}$, and similarly for R. What is remarkable is that when one computes the Casimirs $Q_L = \eta^{ab}L_aL_b$ and $Q_R = \eta^{ab}R_aR_b$, one finds that they are independent of α_L and α_R, and we get,

$$Q_L = -p^2 + \hbar^2\,, \qquad Q_R = -p^2 + \hbar^2\,. \tag{3.28}$$

However to do this the result $\mathrm{tr}(\tilde{\mathsf{u}} p \mathsf{T}_2 \tilde{\mathsf{u}}^{-1}) = -2i\hbar$ is needed, which appears only to be calculable using canonical coordinates (we used the Cartan coordinates to do this). Note that to show equations (3.27) the explicit form of the braiding matrix (3.25) is required. Straightforward manipulations lead to,

$$[\mathsf{L}_1, \mathsf{u}_2] = \mathsf{u}_1\mathsf{u}_2[\mathsf{B}_{12}^{-1} - \mathsf{I}\otimes\mathsf{I}, p(\mathsf{T}_2)_1]\mathsf{u}_1^{-1} + i\hbar\mathsf{u}_1\mathsf{u}_2(\mathsf{T}_2\otimes\mathsf{T}_2)\mathsf{B}_{12}^{-1}\mathsf{u}_1^{-1} \tag{3.29}$$

and from equation (3.25) it is easily verified that,

$$\begin{aligned} [\mathsf{B}_{12}^{-1} - \mathsf{I}\otimes\mathsf{I}, p(\mathsf{T}_2)_1] &= -i\hbar(\mathsf{T}_1\otimes\mathsf{T}_1 + \mathsf{T}_3\otimes\mathsf{T}_3) \\ (\mathsf{T}_2\otimes\mathsf{T}_2)(\mathsf{B}_{12}^{-1} - 1\otimes 1) &= \mathsf{B}_{12}^{-1} - \mathsf{I}\otimes\mathsf{I} \end{aligned} \tag{3.30}$$

which when substituted into (3.29) give,

$$[\mathsf{L}_1, \mathsf{u}_2] = -2i\hbar\mathsf{u}_1\mathsf{u}_2\mathbb{P}\mathsf{u}_1^{-1} + i\hbar\mathsf{u}_2\,. \tag{3.31}$$

Now using equation (2.20) where $\mathcal{A} = \mathrm{End}\,\mathcal{H}$ we get the first of equation (3.27) as required, by the sole use of the quantum conditions (3.16), (3.17) and (3.25).

4. Wavefunctions and comparison to QFT

It is interesting to see how the quantum mechanics derived compares to QFT in AdS_3 which, of course, has been well studied, for example in [11]. More precisely, we will compare the propagator in quantum mechanics to the Wightman function in the QFT.

In the quantum mechanics we look for position eigenstates $|g\rangle$ satisfying $\hat{g}|g\rangle = g|g\rangle$, where we have put a hat on the operator for clarity. Note that it is a nontrivial fact that we can simultaneously diagonalise the matrix elements of $\hat{g}$, but in fact one can, since they all commute amongst each other. This is the locality condition addressed in the "Chiral quantization" section. The quantity we are interested in then is the amplitude $\langle g|g'\rangle$. This can be computed using any convenient basis $\{|i\rangle\}$ for the Hilbert space $\mathcal{H}$, as $\langle g|g'\rangle = \sum_i \langle g|i\rangle\langle i|g'\rangle$. One could use the basis we mentioned earlier $|\mu, m, \bar{m}\rangle$ and compute the matrix elements of the operator matrix $\hat{g}$; in principle this would allow one to deduce $\langle \mu, m, \bar{m}|g\rangle$. It would be nice to perform such a calculation, however instead we take a shortcut. Note that [14] has addressed the position representation starting from the Klein-Gordon equation. In a position representation we seek eigenfunctions of the current algebra which can be represented by the differential operators,

$$
\begin{aligned}
L_2 &= i\hbar\frac{\partial}{\partial v}\,, \qquad R_2 = -i\hbar\frac{\partial}{\partial u}\,,\\
L_\pm &= i\hbar e^{\mp 2iv}\left(\coth 2\rho\frac{\partial}{\partial v} - \operatorname{cosech} 2\rho\frac{\partial}{\partial u} \pm i\frac{\partial}{\partial \rho}\right),\\
R_\pm &= i\hbar e^{\pm 2iu}\left(\coth 2\rho\frac{\partial}{\partial u} - \operatorname{cosech} 2\rho\frac{\partial}{\partial v} \mp i\frac{\partial}{\partial \rho}\right).
\end{aligned}
\tag{4.1}
$$

It is straightforward to check that they satisfy the correct algebra

$$
[L_a, L_b] = 2\hbar i\epsilon_{ab}{}^c L_c\,, \qquad [R_a, R_b] = 2i\hbar\epsilon_{ab}{}^c R_c\,, \qquad [L_a, R_b] = 0\,,
\tag{4.2}
$$

where $L_\pm = L_3 \pm iL_1$. The Casimirs $Q_L = \eta^{ab}L_aL_b$ and Q_R, turn out to be

$$
Q_L = Q_R = -\hbar^2\Box\,,
\tag{4.3}
$$

where $\Box$ is the laplacian on AdS_3. Thus in an irreducible representation the eigenfunctions of L_2, R_2, Q satisfy the Klein-Gordon equation with a mass that depends on which expression for the Casimir is used. Therefore the propagator $\langle g|g'\rangle$ will be the same function as the Wightman function for the QFT with a suitable replacement for the mass parameter. The Wightman function G for a scalar field satisfying $(\Box - \xi)\phi = 0$, is a function of the invariant distance $\sigma(x, x') = \frac{1}{2}\eta^{(4)}_{\mu\nu}(x - x')^\mu(x - x')^\nu$ (x is the coordinate in the embedding space $\mathbb{R}^{2,2}$ corresponding to the group element g) , and is given by [11],[6]

$$
G(z) = \frac{1}{4\pi}(z^2 - 1)^{-1/2}\left[z + (z^2 - 1)^{-1/2}\right]^{1-\lambda}\,,
\tag{4.4}
$$

[6] They provide the expression on the universal cover of AdS_3, but actually it is the same for AdS_3, since the expression is periodic in t.

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where $z = 1 + \sigma + i\epsilon \,\mathrm{sgn}[\sin(t - t')]$ and $\lambda = 1 + \sqrt{1+\xi}$ for $\xi \geq 0$ and $\xi = -1$, and $\lambda = 1 \pm \sqrt{1+\xi}$ for $-1 < \xi < 0$. Thus we end up with an expression for $\langle g|g'\rangle$ which equals $G(z)$ with $\xi = \mu^2 - 1$ for $Q = -p^2 + \hbar^2$ and $\xi = \mu^2$ for $Q = -p^2$. It is interesting to note that $\xi = \mu^2 - 1$ appears to correspond to some sort of critical coupling since $\lambda = 1 + \mu$ in this case. It should be noted that the well-known Brietenlohner-Freedman bound for the Klein-Gordon equation in AdS_3 is $\xi \geq -1$ (in these conventions) [15, 1]; this is consistent with $\xi = \mu^2 - 1$ which corresponds to the Casimir $Q = -p^2 + \hbar^2$. Hence it appears that the quantum mechanics with the quantum correction to the Casimir is consistent with the field theory, despite the fact that the definition of mass is somewhat arbitrary.

5. Conclusions

We studied the motion of a free massive particle moving on the group manifold AdS_3 both classically and quantum mechanically in a covariant canonical formalism. We derived two different quantum theories depending on whether the constraint that the motion be future directed is imposed classically or quantum mechanically. The allowed values of the mass of a particle quantum mechanically are quantized in either case. Only certain representations of the current algebra were allowed in the Hilbert space, namely (most of) the discrete series. Interestingly in the unconstrained theory the continuous series in the exceptional interval is present leading to a small continuum of mass states; these states have the property that they can flip between moving forward or backward in time, and of course have no classical counterpart.

A quantization in terms of "chiral" variables of the theory was carried out, which amounts to determining something called a braiding matrix, and this gave the same results as canonical quantization of the unconstrained theory.

Upon comparison with QFT we found that one of the quantum theories corresponds to a critical coupling (neither minimal or conformal), which is consistent with the Breitenlohner-Freedman bound. It would also be interesting to see the precise role of boundary conditions, which must feature since AdS_3 is not globally hyperbolic,[7] in the quantum mechanics.

Possible extensions of this work include a similar treatment for massless particles, a more systematic derivation of the position representation, and of course attacking the problem of a canonical quantization of string theory in AdS_3.

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[7]These correspond to the two different values of λ for $-1 < \xi < 0$.

A. Some representation theory

In this section we summarise the unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R})$ and its universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, see [17, 1]. Both have the same algebras, namely $sl(2,\mathbb{R})$. Let $\{t_a : a = 1, 2, 3\}$ be a basis for the algebra satisfying $[t_a, t_b] = \epsilon_{ab}{}^c t_c$, where $\epsilon_{ab}{}^c = \epsilon_{abd}\eta^{dc}$ and η is the diagonal metric with $\eta_{11} = \eta_{22} = -\eta_{33} = 1$. . Consider a unitary representation $R : t_a \mapsto -iJ_a$, so $J_a^\dagger = J_a$. Therefore we have,

$$[J_a, J_b] = i\epsilon_{ab}{}^c J_c\,. \tag{A.1}$$

Define $J_\pm = J_1 \pm iJ_2$. The Casimir operator is $Q = \eta_{ab}J_aJ_b = J_1^2 + J_2^2 - J_3^2$. Now, lets examine the spectrum of J_3 in an irreducible unitary representation; necessarily the Casimir will be diagonal.

$$\begin{aligned} Q|j,m\rangle &= -j(j+1)|j,m\rangle \\ J_3|j,m\rangle &= m|j,m\rangle\,, \end{aligned} \tag{A.2}$$

where $j \equiv -\frac{1}{2} - \sqrt{\frac{1}{4} - q}$ and q is the eigenvalue of Q. Since J_a is hermitean, $\{|j,m\rangle\}$ can be chosen to be an orthonormal basis for the carrier space to the representation, also m must be real. It is easy to show that,

$$J_\pm|j,m\rangle = \sqrt{m(m\pm 1) - j(j+1)}\,|j,m\pm 1\rangle\,. \tag{A.3}$$

So far everything stated applies equally to $\mathrm{SL}(2,\mathbb{R})$ and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The differences arise from the differences in the spectrum of J_3. By considering the representation induced on the enveloping algebra it is easy to show that m is either integer or half-integer in the case of $\mathrm{SL}(2,\mathbb{R})$, whereas this restriction fails for $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

1. The principal discrete representations correspond to highest and lowest weight representations. More explicitly, D_j^+ is defined by $J_-|j,-j\rangle = 0$, and thus $m = -j, -j+1, \ldots$ and $j < 0$. Similarly D_j^- is defined by $J_+|j,j\rangle = 0$, and therefore $m = j, j-1, \ldots$ and $j < 0$. For $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ j is not restricted further, whereas for $\mathrm{SL}(2,\mathbb{R})$ $2j \in -\mathbb{N}$.

2. The principal continuous representations correspond to no highest or lowest weight, so the spectrum of J_3 is unbounded. Also, $j = -\frac{1}{2} + i\kappa$ and $m = \alpha, \alpha \pm 1, \ldots$, where $\kappa \in \mathbb{R} - \{0\}$ and $0 \leq \alpha < 1$. These representations are labelled by C_j^α, and $\forall \alpha \in [0,1)$ correspond to irreducible representations of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and for $\alpha = 0, \frac{1}{2}$ gives two inequivalent irreducible representations of $\mathrm{SL}(2,\mathbb{R})$.

3. There is another set of representations for which the spectrum of J_3 is unbounded. These are called the exceptional representations, for which $-1 < j < -\frac{1}{2}$ $(0 < q < 1/4)$ and $m = \alpha, \alpha \pm 1, \ldots$, and will be denoted by E_j^α. Note these are often also called C_j^α or C_q^α, and there is no confusion with the representations above since we have different values of j. Here, $\alpha = 0$ corresponds to representations of $\mathrm{SL}(2,\mathbb{R})$.

B. Reparameterisation invariance at the quantum level

Here we provide justification for why gauge fixing the lagrangian $\mathcal{L} = \frac{1}{e}\dot{x}^2 - m^2 e$ by letting $e = 1$ does not spoil reparameterisation invariance quantum mechanically; see also [16] for a different argument. So to begin we note that the lagrangian is singular, leading to the primary constraint $\phi_1 = p_e \equiv \frac{\partial \mathcal{L}}{\partial \dot{e}} \approx 0$. We want to also impose the constraint $\phi_2 = e - 1 \approx 0$. If we look for secondary constraints we find $\phi_3 = \frac{1}{4}\pi^2 + m^2 \approx 0$, where π_μ is conjugate to x^μ, and no more occur. It is easy to see that ϕ_3 is first-class, and ϕ_1, ϕ_2 are second class. Now, we replace Poisson brackets on the phase space with canonical variables (x, π, e, p_e), with the Dirac bracket defined by $\{A, B\}_D = \{A, B\} - \{A, \phi_a\} C^{-1}_{ab} \{\phi_b, B\}$ for any two observables A, B, where $C_{ab} = \{\phi_a, \phi_b\}$ and $a, b \in \{1, 2\}$. Now, using the Dirac brackets instead of the Poisson brackets, allows us to set ϕ_1 and ϕ_2 strongly to zero; therefore $\{A, \phi_a\} = 0$ for any observable A, since once the second class constraints are set strongly to zero A will only be a function of (x, π). This allows us to deduce that $\{A, B\}_D = \{A, B\}$. Hence the lagrangian $\mathcal{L} = \dot{x}^2$ (the constant m^2 isn't important and will be dropped) with the constraint ϕ_3 will give rise to the same phase space and equations of motion as the manifestly reparameterisation invariant lagrangian. Also, the constraint is easy to implement quantum mechanically giving rise to the BRST operator $Q = \phi_3 c$ which is nilpotent (since c is a ghost). The physical state conditions are $Q|\psi\rangle = 0$ and $b|\psi\rangle = 0$, where b is the antighost and $\{b, c\} = 1$; this implies $\phi_3|\psi\rangle = 0$, which becomes $p^2|\psi\rangle = m^2|\psi\rangle$.

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